STEADY TEMPERATURE FIELD IN THE ELEMENTS OF OPTICAL SYSTEMS

S. V. Tikhonov

UDC 535.211

We study an approximate method for the analytic determination of steady temperature fields in the elements of optical systems. The method assumed can be used to study temperature fields of other objects with a curvilinear boundary.

Optical systems often undergo the action of various energetic factors. This leads to the appearance of inhomogeneous temperature fields in the system and thermoelastic stresses in its separate elements (lenses, illuminators, mirrors). The presence of stresses produces deformations in optical elements, changes their form, and the parameters of the system are different from those calculated. This is expressed as a transform of thermooptical aberrations. In this connection we are most interested in the initial determination of the effect of energetic factors on the quality of the optical system operation in order to further use this data in the planning stage for developing compensation systems or automatic control. In this scheme the calculation of the temperature field is the first stage which imposes definite requirements on the solution, notably the sufficient accuracy and the relative simplicity of the final result. Below we study an approximate analytic method to determine steady temperature fields in optical elements which satisfies these requirements to a sufficient degree.

General Formulation of the Problem

We study a lens whose surfaces S_1, S_2, and S_3 are located in three media with the different temperatures t_{C1}, t_{C2}, and t_{C3} (Fig. 1). We assume that the heat exchange of the surfaces with the media is realized according to Newton's law with the constant coefficients of heat exchange α_1, α_2, and α_3. On the lens surface we give the heat flows with surface density $q_1(s_1)$, $q_2(s_2)$, $q_3(s_3)$ that can be a coordinate function in the general case. Internal energy sources with volume power $w(v)$ can also act on the lens. The absorbed part of the falling flows can play the role of the sources. Both the convex and the concave refracting surfaces S_1 and S_2 in the cylindrical coordinate system are described by the equations $z(s_1) = f_1(r)$, $z(s_2) = f_2(r)$. To solve the problem we assume that the physical parameters are constant. Assuming axial symmetry, we write the mathematical notation of the problem formulated as follows:

$$\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{\partial^2 t}{\partial z^2} + \frac{w(r,\ z)}{\lambda} = 0, \tag{1}$$

$$\left[\frac{\partial t}{\partial n_1} + \frac{\alpha_1}{\lambda}(t - t_{C1})\right]_{z=f_1} = \frac{q_1(r)}{\lambda}, \tag{2a}$$

$$\left[\frac{\partial t}{\partial n_2} + \frac{\alpha_2}{\lambda}(t - t_{C2})\right]_{z=f_2} = \frac{q_2(r)}{\lambda}, \tag{2b}$$

$$\left[\frac{\partial t}{\partial r} + \frac{\alpha_3}{\lambda}(t - t_{C3})\right]_{r=R} = \frac{q_3(z)}{\lambda}. \tag{2c}$$

We study the case of constant energetic actions

$$q_1(r) \equiv q_{10};\ q_2(r) \equiv q_{20};\ q_3(r) \equiv q_{30};\ w(r,\ z) \equiv w_0.$$

This limitation is made to illustrate the method assumed more clearly and has no principal value.

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 29, No. 5, pp. 884-891, November, 1975. Original article submitted July 2, 1974.

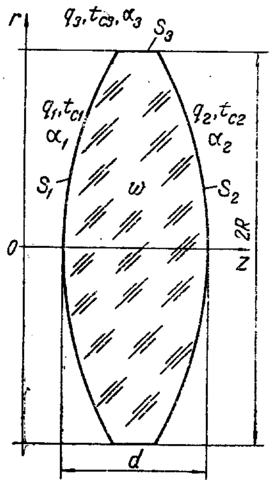

Fig. 1. Thermal model of problem.

We write the temperature derivative along the normal which enters boundary conditions (2a) and (2b) by means of the equations

$$\frac{\partial t}{\partial n} = \frac{\partial t}{\partial z}\cos(n,\ z) + \frac{\partial t}{\partial r}\cos(n,\ r). \tag{3}$$

Using the axial thickness of the lens d and the radius R, we transfer to the relative coordinates $\rho = r/R$, $\bar{z} = z/d$. Here the parameter $\kappa = d^2/R^2$ appears in problem (1)-(2). This quantity is usually small for most lenses, which allows us to use the excitation method [1] and select κ as the small parameter ε. We also note that in optics the main interest is the superheating $\vartheta = t - t_{C3}$, with respect to which problem (1)-(2) takes the following form with the small parameter taken into account:

$$\varepsilon\frac{\partial^2\vartheta}{\partial\rho^2} + \frac{\varepsilon}{\rho}\frac{\partial\vartheta}{\partial\rho} + \frac{\partial^2\vartheta}{\partial\bar{z}^2} + \varepsilon\frac{w_0R^2}{\lambda} = 0, \tag{4}$$

$$\left[\varepsilon\bar{f}_1'\frac{\partial\vartheta}{\partial\rho} - \frac{\partial\vartheta}{\partial\bar{z}} + \mathrm{Bi}_1\sqrt{1+\varepsilon(\bar{f}_1')^2}\vartheta\right]_{\bar{z}=\bar{f}_1} = Q_{10}\sqrt{1+\varepsilon(\bar{f}_1')^2}, \tag{5a}$$

$$\left[-\varepsilon\bar{f}_2'\frac{\partial\vartheta}{\partial\rho} + \frac{\partial\vartheta}{\partial\bar{z}} + \mathrm{Bi}_2\sqrt{1+\varepsilon(\bar{f}_2')^2}\,\vartheta\right]_{\bar{z}=\bar{f}_2} = Q_{20}\sqrt{1+\varepsilon(\bar{f}_2')^2}, \tag{5b}$$

$$\left[\sqrt{\varepsilon}\frac{\partial\vartheta}{\partial\rho} + \mathrm{Bi}_3\vartheta\right]_{\rho=1} = Q_{30}. \tag{5c}$$

Here

$$\bar{f}_1 = \frac{f_1}{d};\ \bar{f}_2 = \frac{f_2}{d};\ \mathrm{Bi}_j = \frac{\alpha_j d}{\lambda},\ j = 1,\ 2,\ 3.$$

The quantities

$$Q_{10} = \frac{q_{10}d}{\lambda} + (t_{C1} - t_{C3})\,\mathrm{Bi}_1,\ Q_{20} = \frac{q_{20}d}{\lambda} + (t_{C2} - t_{C3})\,\mathrm{Bi}_2,\ Q_{30} = \frac{q_{30}d}{\lambda}$$

can be studied as generalized flows.

The linear formulation of the problem allows us to study the effect of each of the energetic actions w_0, Q_{10}, Q_{20}, Q_{30} separately. Here the desired temperature field is obtained by means of the simple summation of the separate solutions.

Internal Energy Source

The temperature field of the lens with an internal energy source is described by Eq. (4) under homogeneous boundary conditions (5). In accordance with the excitation method we present the expansion of the desired function ϑ_0 in the central (with respect to ρ) region as follows [1]:

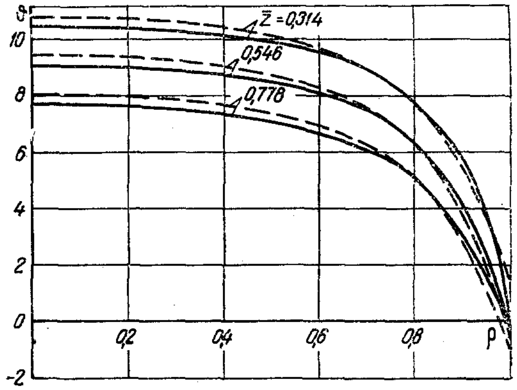

Fig. 2. Distribution of superheating $\vartheta = t - t_{c3}$ in radial direction; solid curves refer to numerical solution; dashed curves, to analytic solution. ϑ, °K.

$$\vartheta_0 = \sum_{K=1}^{\infty} \mu_K(\varepsilon)\varphi_K(\rho, \bar{z}), \quad \frac{\mu_{K+1}(\varepsilon)}{\mu_K(\varepsilon)} \ll 1. \tag{6}$$

The substitution of Eq. (6) into the boundary-value problem and an analysis of the equations obtained allow us to determine the expansion coefficients

$$\mu_K(\varepsilon) = \varepsilon^K.$$

Setting the terms equal that have the identical order of ε, we obtain a sequence of boundary-value problems to determine $\varphi_k(\rho, z)$:

$$\left.\begin{aligned} &\frac{\partial^2 \varphi_1}{\partial \bar{z}^2} + \frac{w_0 R^2}{\lambda} = 0, \\ &\left[-\frac{\partial \varphi_1}{\partial \bar{z}} + \mathrm{Bi}_1 \varphi_1\right]_{\bar{z}=\bar{f}_1} = 0, \quad \left[\frac{\partial \varphi_1}{\partial \bar{z}} + \mathrm{Bi}_2 \varphi_1\right]_{\bar{z}=\bar{f}_2} = 0, \end{aligned}\right\} \tag{7}$$

$$\left.\begin{aligned} &\frac{\partial^2 \varphi_2}{\partial \bar{z}^2} = -\frac{\partial^2 \varphi_1}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial \varphi_1}{\partial \rho}, \\ &\left\{\bar{f}_1' \frac{\partial \varphi_1}{\partial \rho} - \frac{\partial \varphi_2}{\partial \bar{z}} + \mathrm{Bi}_1\left[\varphi_2 + \frac{1}{2}\varphi_1 (\bar{f}_1')^2\right]\right\}_{\bar{z}=\bar{f}_1} = 0, \\ &\left\{-\bar{f}_2' \frac{\partial \varphi_1}{\partial \rho} + \frac{\partial \varphi_2}{\partial \bar{z}} + \mathrm{Bi}_2\left[\varphi_2 + \frac{1}{2}\varphi_1 (\bar{f}_2')^2\right]\right\}_{\bar{z}=\bar{f}_2} = 0 \end{aligned}\right\} \tag{8}$$

and so forth.

The solution of problem (7) can be represented as follows:

$$\varphi_1 = \frac{w_0 R^2}{\lambda}\Phi(\rho, \bar{z}), \tag{9}$$

where

$$\Phi(\rho, \bar{z}) = -\frac{1}{2}\bar{z}^2 + \frac{\frac{1}{2}\mathrm{Bi}_1\mathrm{Bi}_2(\bar{f}_2^2 - \bar{f}_1^2) + \mathrm{Bi}_1\bar{f}_2 + \mathrm{Bi}_2\bar{f}_1}{\mathrm{Bi}_1 + \mathrm{Bi}_2 + \mathrm{Bi}_1\mathrm{Bi}_2(\bar{f}_2 - \bar{f}_1)}\bar{z} - \frac{\frac{1}{2}\mathrm{Bi}_1\mathrm{Bi}_2\bar{f}_1\bar{f}_2(\bar{f}_2 - \bar{f}_1) - (\bar{f}_2 - \bar{f}_1) + \bar{f}_1\bar{f}_2(\mathrm{Bi}_1 + \mathrm{Bi}_2) - \frac{1}{2}(\mathrm{Bi}_1\bar{f}_1^2 + \mathrm{Bi}_2\bar{f}_2^2)}{\mathrm{Bi}_1 + \mathrm{Bi}_2 + \mathrm{Bi}_1\mathrm{Bi}_2(\bar{f}_2 - \bar{f}_1)}. \tag{10}$$

According to the terminology in [1], we call solution (9) the internal solution. We limit ourselves to the first approximation of Eq. (6) and then find the external solution, which is related to the search for function ϑ_0 near $\rho = 1$. To do this we introduce the new variable

$$\rho_0 = \frac{1-\rho}{\delta(\varepsilon)},$$

where $\delta(\varepsilon)$ is a small quantity, and we transfer to this variable in Eq. (4) and in the homogeneous boundary conditions (5):

$$\frac{\varepsilon}{\delta^2}\frac{\partial^2\bar{\vartheta}_0}{\partial\rho_0^2} - \frac{\varepsilon}{\delta(1-\rho_0\delta)}\frac{\partial\bar{\vartheta}_0}{\partial\rho_0} + \frac{\partial^2\bar{\vartheta}_0}{\partial\bar{z}^2} + \varepsilon\frac{w_0R^2}{\lambda} = 0, \tag{11}$$

$$\left[\frac{\varepsilon}{\delta^2}F_1'\frac{\partial\bar{\vartheta}_0}{\partial\rho_0} - \frac{\partial\bar{\vartheta}_0}{\partial\bar{z}} + \mathrm{Bi}_1\sqrt{1+\frac{\varepsilon}{\delta^2}(F_1')^2}\bar{\vartheta}_0\right]_{\bar{z}=F_1} = 0, \tag{12a}$$

$$\left[-\frac{\varepsilon}{\delta^2}F_2'\frac{\partial\bar{\vartheta}_0}{\partial\rho_0} + \frac{\partial\bar{\vartheta}_0}{\partial\bar{z}} + \mathrm{Bi}_2\sqrt{1+\frac{\varepsilon}{\delta^2}(F_2')^2}\bar{\vartheta}_0\right]_{\bar{z}=F_2} = 0, \tag{12b}$$

$$\left[-\frac{\sqrt{\varepsilon}}{\delta}\frac{\partial\bar{\vartheta}_0}{\partial\rho_0} + \mathrm{Bi}_3\bar{\vartheta}_0\right]_{\rho_0=0} = 0. \tag{12c}$$

Here we introduce the following notation:

$$\bar{f}_1(\rho = 1-\rho_0\delta) \equiv F_1(\rho_0),\ F_1' = \frac{dF_1}{d\rho_0};$$

$$\bar{f}_2(\rho = 1-\rho_0\delta) \equiv F_2(\rho_0),\ F_2' = \frac{dF_2}{d\rho_0}.$$

To determine quantity $\delta(\varepsilon)$ we must analyze the behavior of Eq. (11) for various relations of δ and ε. The results of the analysis is the equation $\delta(\varepsilon) = \sqrt{\varepsilon}$. Taking the value found for δ into account, Eq. (11) takes the form

$$\frac{\partial^2\bar{\vartheta}_0}{\partial\rho_0^2} - \frac{\sqrt{\varepsilon}}{1-\rho_0\sqrt{\varepsilon}}\frac{\partial\bar{\vartheta}_0}{\partial\rho_0} + \frac{\partial^2\bar{\vartheta}_0}{\partial\bar{z}^2} + \varepsilon\frac{w_0R^2}{\lambda} = 0. \tag{13}$$

We represent $\bar{\vartheta}_0$ in series form,

$$\bar{\vartheta}_0 = \sum_{K=1}^{\infty}\nu_K(\varepsilon)\bar{\varphi}_K(\rho_0, \bar{z}),\ \frac{\nu_{K+1}(\varepsilon)}{\nu_K(\varepsilon)} \ll 1 \tag{14}$$

and substitute it into boundary conditions (12) and Eq. (13), after initially expanding them in Taylor series term by term in the region of point $\rho = 1$. The analysis of the equations obtained allows us to determine the values of the expansion coefficients,

$$\nu_K(\varepsilon) = \varepsilon^{\frac{K+1}{2}}.$$

By setting equal the terms that have the identical order of ε, we obtain a sequence of boundary-value problems to find the function $\varphi_K(\rho_0, z)$. For the first approximation we have

$$\frac{\partial^2\bar{\varphi}_1}{\partial\rho_0^2} + \frac{\partial^2\bar{\varphi}_1}{\partial\bar{z}^2} + \frac{w_0R^2}{\lambda} = 0, \tag{15}$$

$$\left[-\frac{\partial\bar{\varphi}_1}{\partial\bar{z}} + \mathrm{Bi}_1\bar{\varphi}_1\right]_{\bar{z}=\bar{f}_1(1)} = 0;\ \left[\frac{\partial\bar{\varphi}_1}{\partial\bar{z}} + \mathrm{Bi}_2\bar{\varphi}_1\right]_{\bar{z}=\bar{f}_2(1)} = 0;$$
$$\left[-\frac{\partial\bar{\varphi}_1}{\partial\rho_0} + \mathrm{Bi}_3\bar{\varphi}_1\right]_{\rho_0=0} = 0. \tag{16}$$

We solve the problem given by the averaging method [2], applying the following integral operator to the initial equation (15) and boundary conditions (16):

$$I_{\bar{z}}\ [\bar{\varphi}_1] = \frac{1}{\bar{f}_2(1) - \bar{f}_1(1)} \int\limits_{\bar{f}_1(1)}^{\bar{f}_2(1)} \bar{\varphi}_1 d\bar{z} = \bar{\varphi}_{1\bar{z}}\ .$$

We omit the details of the solutinn, which is presented in [2], and write the final result:

$$\bar{\varphi}_1 = \Phi(\rho = 1,\ \bar{z}) \left[\frac{Cm^3 - \mathrm{Bi}_3 \frac{w_0 R^2}{\lambda} - \mathrm{Bi}_3 m^2 C}{m + \mathrm{Bi}_3} \exp(-m\rho_0) + Cm^2 \exp(m\rho_0) + \frac{w_0 R^2}{\lambda} \right], \tag{17}$$

where

$$m^2 = \frac{\mathrm{Bi}_1 \psi_1 + \mathrm{Bi}_2 \psi_2}{\bar{f}_2(1) - \bar{f}_1(1)}; \tag{18}$$

ψ_1 and ψ_2, the coefficients that characterize the inhomogeneity of the temperature field, are determined by the equations

$$\psi_1 = \frac{\bar{\varphi}_1 [\rho_0,\ \bar{z} = \bar{f}_1(1)]}{\bar{\varphi}_{1\bar{z}}}; \quad \psi_2 = \frac{\bar{\varphi}_1 [\rho_0,\ \bar{z} = \bar{f}_2(1)]}{\bar{\varphi}_{1\bar{z}}}. \tag{19}$$

In the first approximation the coefficients ψ_1 and ψ_2 are assumed to be constant and considered as equal equations of the mean integral values of the numerator and denominator [2].

The indeterminate constant C figures in Eq. (17) due to the fact that there is no single boundary condition to determine problem (15)-(16) fully. The condition of joining the solutions of ϑ_0 and $\bar{\vartheta}_0$, enters its role, and this condition leads to the satisfaction of the boundary condition

$$\lim_{\rho \to 1} \vartheta_0 = \lim_{\rho_0 \to \infty} \bar{\vartheta}_0 = A.$$

In our case this condition is satisfied if C = 0. Consequently,

$$\bar{\varphi}_1 = \frac{w_0 R^2}{\lambda} \Phi(\rho = 1,\ \bar{z}) \left[1 - \frac{\mathrm{Bi}_3}{m + \mathrm{Bi}_3} \exp(-m\rho_0) \right].$$

To obtain an equally suitable solution in the region $\rho \in [0,\ 1]$, we sum the internal and external solutions and subtract the general part A from them. In physical variables we obtain

$$\vartheta_0 = \frac{w_0 d^2}{\lambda} \left\{ \Phi(\rho,\ \bar{z}) - \frac{\mathrm{Bi}_3}{m + \mathrm{Bi}_3} \Phi(\rho = 1,\ \bar{z}) \exp\left[-m(1 - \rho) \frac{R}{d} \right] \right\}. \tag{20}$$

To complete the solution we determine coefficient m from Eqs. (18)-(20)

$$m^2 = \frac{\mathrm{Bi}_1 + \mathrm{Bi}_2 + \mathrm{Bi}_1 \mathrm{Bi}_2 [\bar{f}_2(1) - \bar{f}_1(1)]}{[\bar{f}_2(1) - \bar{f}_1(1)] \left\{ 1 + \frac{1}{3}(\mathrm{Bi}_1 + \mathrm{Bi}_2) [\bar{f}_2(1) - \bar{f}_1(1)] + \frac{1}{12} \mathrm{Bi}_1 \mathrm{Bi}_2 [\bar{f}_2(1) - \bar{f}_1(1)]^2 \right\}}. \tag{21}$$

Thus the temperature field of the lens heated by an internal source is described in the first approximation by Eqs. (20), (21), and (10).

The temperature fields produced by the action of other energetic factors are found by a similar scheme. After generalizing the separate results, we can write the final calculated equation that describes in the first approximation the steady temperature field of an element of the optical system subject to complex heating action,

$$t = t_{C3} + \frac{w_0 d^2}{\lambda} \Phi(\rho,\ \bar{z}) + \frac{Q_{10} + Q_{20} + Q_{10} \mathrm{Bi}_2 (\bar{f}_2 - \bar{z}) + Q_{20} \mathrm{Bi}_1 (\bar{z} - \bar{f}_1)}{\mathrm{Bi}_1 + \mathrm{Bi}_2 + \mathrm{Bi}_1 \mathrm{Bi}_2 (\bar{f}_2 - \bar{f}_1)} -$$

$$- \Phi(\rho = 1,\ \bar{z}) \left\{ \frac{w_0 d^2}{\lambda} \frac{\mathrm{Bi}_3}{m + \mathrm{Bi}_3} \exp\left[-m(1 - \rho) \frac{R}{d} \right] - \right.$$

$$- \frac{m^2 Q_{30}}{m + \mathrm{Bi}_3} \exp\left[-m(1 - \rho) \frac{R}{d} \right] + \frac{Q_{10} \mathrm{Bi}_3}{(p + \mathrm{Bi}_3)[\bar{f}_2(1) - \bar{f}_1(1)]} \times$$

$$\times \exp\left[-p(1-\rho)\frac{R}{d}\right] + \frac{Q_{20}\mathrm{Bi}_3}{(s+\mathrm{Bi}_3)[\bar{f}_2(1)-\bar{f}_1(1)]}\exp\left[-s(1-\rho)\frac{R}{d}\right]\Bigg\}, \tag{22}$$

$$p^2 = \frac{\mathrm{Bi}_1+\mathrm{Bi}_2+\mathrm{Bi}_1\mathrm{Bi}_2[\bar{f}_2(1)-\bar{f}_1(1)]}{[\bar{f}_2(1)-\bar{f}_1(1)]\left\{1+\frac{1}{2}\mathrm{Bi}_2[\bar{f}_2(1)-\bar{f}_1(1)]\right\}},$$

$$s^2 = \frac{\mathrm{Bi}_1+\mathrm{Bi}_2+\mathrm{Bi}_1\mathrm{Bi}_2[\bar{f}_2(1)-\bar{f}_1(1)]}{[\bar{f}_2(1)-\bar{f}_1(1)]\left\{1+\frac{1}{2}\mathrm{Bi}_1[\bar{f}_2(1)-\bar{f}_1(1)]\right\}}.$$

We estimate the error of the solutions determined. For this we compare the approximate solution (22) with the one calculated on the example of a biconvex lens with the parabolic refracting surfaces S_1 and S_2 given by the functional equations

$$f_1 = d_1\rho^2,\ f_2 = d + d_2\rho^2;\ d_1 = 0.02981\ \mathrm{m},\ d_2 = -0.02112\ \mathrm{m},\ d = 0.095\ \mathrm{m},\ R = 0.35\ \mathrm{m}.$$

We give the energetic action by the flow falling on the front surface of the lens S_1,

$$q_{10} = 76.0\ \mathrm{W/m^2},\ q_{20} = 0,\ q_{30} = 0;\ w_0 = 126.3\ \mathrm{W/m^3},$$

and we assume the temperatures of the three media to be identical. Finally, we give the conditions of heat exchange,

$$\alpha_1 = 3\ \mathrm{W/m^2\cdot{}^\circ K},\ \alpha_2 = 7.7\ \mathrm{W/m^2\cdot{}^\circ K},\ \alpha_3 \to \infty$$

and the thermal-conductivity coefficient $\lambda = 0.72$ W/m•°K. The calculation, with which the analytic function was compared, is conducted according to the network method with the number of nodes about 400. The results of comparing the solutions for the superheating ϑ in the radial direction are presented in Fig. 2. We see from the graphs that for $\rho \leq 0.9$ the numerical and analytic solutions are distinguished by not more than 4%, i.e., at the margin of error for the numerical solution. For $0.9 < \rho \leq 1$ the relative error is considerably higher. However, the relative error already characterizes the accuracy of the solutions poorly on such a small interval of the variation ρ and on the comparatively low-temperature level. The disparity of the solutions in this range does not exceed 14% of the maximum superheating. In the case of finite values of the heat-exchange coefficient α_3, the curvature of the temperature field in the radial direction will be less and we should expect a better agreement of the solutions.

We note, in conclusion, that the region $\rho < 0.8$ is of the greatest interest in studying problems in optics, since the remaining part of the lens is not operative. Thus, because of the principle of local effect, the errors omitted in determining the temperatures at interval $0.9 < \rho \leq 1$ do not produce significant errors in determining the temperature stresses in the region of practical interest.

NOTATION

n_1, n_2, directions of external normals to surfaces S_1 and S_2 R, lens radius; d, its axial thickness; t(r, z), lens temperature at point (r, z); $\rho = r/R$, $\bar{z} = z/d$, relative lens dimensions.

LITERATURE CITED

1. J. Cole, Perturbation Methods in Applied Mathematics, Blaisdell, Waltham, Massachusetts (1968).
2. A. Akaev and G. N. Dul'nev, Inzh.-Fiz. Zh., 22, No. 4 (1972).